Phys 410
Fall 2015
Lecture #8 Summary
24 September, 2015

We considered un-driven damped oscillations produced by a damping force that is linear in velocity $m\ddot{x} + b\dot{x} + kx = 0$. Divide the mechanical equation through by mass m and define two important rates: $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$, where $2\beta \equiv b/m$, and $\omega_0^2 \equiv k/m$. We tried a solution of the form $x(t) = e^{rt}$, and found an auxiliary equation with solution $r = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$. The general solution is $x(t) = e^{-\beta t}\left[C_1 e^{\sqrt{\beta^2-\omega_0^2}\ t} + C_2 e^{-\sqrt{\beta^2-\omega_0^2}\ t}\right]$. The form of the solution depends critically on the relative size of the two rates β and ω_0.

1) Un-damped oscillator $\beta = 0$. The radical becomes $\sqrt{-\omega_0^2} = i\sqrt{\omega_0^2} = i\omega_0$, and the solution reverts to our previous results $x(t) = C_1 e^{i\omega_0\ t} + C_2 e^{-i\omega_0\ t}$.
2) Weak damping ($\beta < \omega_0$, underdamping). The radical also produces a factor of "i", resulting in $x(t) = e^{-\beta t}\left[C_1 e^{i\omega_1 t} + C_2 e^{-i\omega_1 t}\right]$, with $\omega_1 \equiv \sqrt{\omega_0^2 - \beta^2}$ a frequency lower than the un-damped natural frequency. This equation describes oscillatory motion under an exponentially damped envelope. The damping rate (or decay parameter) is β. One can re-write the solution as $x(t) = A\, e^{-\beta t}\cos(\omega_1 t - \delta)$.
3) Strong damping ($\beta > \omega_0$, overdamping). In this case $\sqrt{\beta^2 - \omega_0^2}$ is real and the solution is $x(t) = C_1 e^{-\left(\beta - \sqrt{\beta^2-\omega_0^2}\right)\ t} + C_2 e^{-\left(\beta + \sqrt{\beta^2-\omega_0^2}\right)\ t}$. This is a sum of two negative exponentials, one of which decays faster than the other – there is no oscillation. The dominant decay parameter is $\beta - \sqrt{\beta^2 - \omega_0^2}$.
4) Critical damping ($\beta = \omega_0$). We need the other independent solution $te^{-\beta t}$ for the second order differential equation. Therefore, the solution is $x(t) = C_1 e^{-\beta t} + C_2 t e^{-\beta t}$. There is also no oscillation, and the decay parameter is β.

If we plot the decay parameter vs. β, the maximum is at the critical damping. This means the oscillation dies out the fastest at the critical damping value. One application of this is in the design of weight scales, and shock absorbers for cars.

We next considered a *driven* damped harmonic oscillator. We take the driving function to be harmonic in time at a new frequency called simply ω, which is an independent quantity from the natural frequency of the un-damped oscillator, called ω_0. The equation of motion is now $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0\cos(\omega t)$. We now employ a trick similar to that used to solve for the velocity of a charged particle in a uniform magnetic field. Consider the complementary problem of the same damped oscillator being driven by a force 90° out of phase, with solution $y(t)$:

$\ddot{y} + 2\beta\dot{y} + \omega_0^2 y = f_0 \sin(\omega t)$. Now define a complex combination of the two unknown functions $z(t) = x(t) + iy(t)$. Combine the two equations in the form of "x-equation" + i "y-equation". This can be written more succinctly as $\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = f_0 e^{i\omega t}$. Note that the solution to the original problem can be found from $x(t) = Re[z(t)]$.

We now want to solve this equation: $\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = f_0 e^{i\omega t}$. We tried a solution of the form $z(t) = Ce^{i\omega t}$ and found this expression for the complex pre-factor: $C = \frac{f_0}{\omega_0^2 - \omega^2 + i2\beta\omega}$. We can write this complex quantity as a magnitude and phase as $C = Ae^{-i\delta}$, where A is the amplitude and δ is the phase, both real numbers. Solving for A and δ in terms of the oscillator parameters gives $A^2 = \frac{f_0^2}{\left(\omega_0^2 - \omega^2\right)^2 + (2\beta\omega)^2}$, and $\delta = \tan^{-1}\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right)$. Finally, we can write the solution to the "z equation" as $z(t) = Ce^{i\omega t} = Ae^{i(\omega t - \delta)}$.

The answer to the original problem is just the real part of this expression: $x(t) = Re[z(t)] = A\cos(\omega t - \delta)$, where ω is the frequency of the driving force. This represents the long-time persistent solution of the motion. It shows that the oscillator eventually adopts the same frequency as the driving force. In addition there is a solution to the homogeneous problem $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$, which we solved before: $x_h(t) = e^{-\beta t}\left[C_1 e^{\sqrt{\beta^2 - \omega_0^2}\ t} + C_2 e^{-\sqrt{\beta^2 - \omega_0^2}\ t}\right]$. The homogeneous solutions represent what the oscillator "wants to do" on its own, but will die out in time. The full solution is the sum of the particular solution and the homogeneous solution. In the case of small loss ($\beta < \omega_0$) the full solution can be written as $x(t) = A\cos(\omega t - \delta) + A_{tr}\, e^{-\beta t} \cos(\omega_1 t - \delta_{tr})$, where the first part is the particular solution and the second part is the transient (homogeneous) solution. We call it transient because of the $e^{-\beta t}$ factor, which shows that the initial motion and initial conditions (specified by A_{tr} and δ_{tr}) will eventually die off and the persistent driven motion will dominate.

The amplitude function $A = \frac{f_0}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + (2\beta\omega)^2}}$ shows a resonant response. As a function of natural frequency ω_0 at fixed driving frequency ω, the maximum amplitude is at $\omega_0 = \omega$. One example is tuning the natural frequency of the local oscillator (LO) in a radio to match the driving frequency from a specific radio station. As a function of frequency ω at fixed natural frequency ω_0, there is a maximum amplitude of the persistent motion response when the driving frequency is equal to $\omega_2 = \sqrt{\omega_0^2 - 2\beta^2}$. The quality factor of the resonance is a measure of how large and sharply peaked the amplitude response looks. It is defined as the ratio of the frequency at which there is peak energy (or power) amplitude over the frequency bandwidth known as the full-width at half maximum (FWHM). The FWHM is defined as the frequency width at the half-power height. (By "power" we mean something proportional to amplitude squared.) The quality factor, or Q, is given by $Q = \omega_0/2\beta$. As the dissipation (parameterized by β) decreases, the quality factor grows. The quality factor is equal to π times the ratio of the decay time $(1/\beta)$ to the natural oscillation period. A high-Q oscillator is therefore one which executes many oscillations on the time scale of the 1/e amplitude decay time.

The phase evolution through resonance goes from 0 well below resonance to π well above resonance, with $\delta = \pi/2$ exactly at resonance. The slope of $\delta(\omega)$ at resonance is $\frac{1}{\beta} = 2Q/\omega_0$.

We considered several examples of resonant phenomena in mechanical and electrical systems, as noted in the Supplementary Material (Lectures 8 and 9). One interesting example was that of crowd synchrony on the Millennium bridge in London. The pedestrians on the bridge acted as a set of periodic driving forces on the bridge position. The bridge acted back on the pedestrians in a manner that caused their motion to synchronize and amplify the oscillations of the bridge. This led to closing of the bridge, and modifications to the structure to increase the damping force on the bridge.